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Original article

On one problem of the plane theory of elasticity for a circular domain with a rectangular hole

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Abstract

The paper considers a plane problem of elasticity for a circle with a rectangular hole. To find a solution, the use is made both of the method of conformal mappings and of boundary value problems of analytic functions. In particular, relying on the well-known Kolosov–Muskhelishvili's formulas, the problem formulated with respect to unknown complex potentials is reduced to the two Riemann–Hilbert problems for a circular ring, and the solutions of the latter problems allow us to construct potentials effectively (analytically). The estimates of the obtained results in the neighborhood of angular points are given. Analogous results (as a particular case) are obtained for a circular domain with a rectilinear cut.

Keywords: Conformal mapping; Kolosov–Muskhelishvili's formulas; The Riemann–Hilbert problem for a circular ring

Statement of the Problem. Let S be a doubly-connected domain occupied by a plate on the plane $z = x + iy$ of a complex variable, bounded by circumference $L_0 = \{|z| = R_0\}$ and rectangle $B_1B_2B_3B_4$ whose sides are parallel to the coordinate axes. By L_1 we denote the boundary of the rectangle (that is, $L_1 = \cup_{k=1}^{4} L_k^{(1)}$, $L_k^{(1)} = B_kB_{k+1}$, $k = \overline{1,4}$, $B_5 = B_1$) and assume that the sides B_1B_2 and B_3B_4 (parallel to the ox-axis) are under the action of constant, normal compressive forces with the given principal vector P (or normal displacements $v_n(t) = v_n^{(k)} = \text{const}$, $t \in L_k^{(1)}$, $k = \overline{1,4}$ are given), and the rest of the boundary $L = L_0 \cup L_1$ is free from the external forces.

Note that certain simplifications in the statement of the problem concerning the cut forms and external forces are insignificant and motivated only to make the problem more clear, namely, to find elastic equilibrium of the plate for a finite doubly-connected domain.

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Analogous problems of the plane theory of elasticity and plate bending for finite doubly-connected domains bounded by polygons have been considered in [1,2].

Solution of the Problem. As is known, the more effective ways of solving the boundary value problems of the plane theory of elasticity by the methods of complex analysis are based on the construction of a conformally mapping function of the given domain onto canonical domains (circle, circular ring). Therefore the above-mentioned methods are little-suited for the effective solution of problems in multi-connected domains. Nevertheless, for some practically important classes of multi-connected domains one manages to construct effectively (analytically) the conformally mapping function of that domain onto a circular ring. These classes involve doubly-connected domains bounded by polygons and their modifications (polygonal domain with a circular hole, or a circle with a polygonal hole). Moreover, the Kolosov–Muskhelishvili's methods in the above-mentioned case allow one to decompose these problems (with respect to complex potentials $\varphi(z)$ and $\psi(z)$) into two Riemann–Hilbert problems for a circular ring, and by solving the latter problems to construct unknown potential in analytical form.

Here we present some results (see [3]) dealing with conformal mapping of a doubly-connected domain, bounded by a polygon, onto a circular ring.

(1) The Dirichlet Problem for a Circular Ring. Let $\mathcal{D}(1 < |z| < R)$ be a circular ring bounded by circumferences $\ell_0(|z| = R)$ and $\ell_1(|z| = 1)$. We consider the problem: find a holomorphic in the ring $\mathcal{D}$ function $\varphi_*(z) = u + iv$ under the boundary condition

$$\mathrm{Re}[\varphi_*(t)] = f_j(t), \quad t \in \ell_j,\ j = 0, 1. \tag{1}$$

The necessary and sufficient condition for solvability of problem (1) is of the form

$$\int_0^{2\pi} f_0(t)d\vartheta = \int_0^{2\pi} f_1(t)d\vartheta \tag{2}$$

and a solution itself is given by the formula

$$\varphi_*(z) = \frac{1}{\pi i}\sum_{j=-\infty}^{\infty}\left[\int_{\ell_0}\frac{f_0(t)}{t-R^{2j}z}\,dt + \int_{\ell_1}\frac{f_1(t)}{t-R^{2j}z}\,dt\right] + ik_1,$$

where k_1 is an arbitrary real constant. Integration on ℓ_0 and ℓ_1 taken in the positive direction leaves the domain D at the left.

(2) Conformal Mapping of a Doubly-Connected Domain, Bounded by Polygons, onto a Circular Ring. Let S^0 be the doubly-connected domain on the plane z of a complex variable, bounded by convex polygons (A) and (B). Assume that (A) is an outer and (B) is an interior boundary of the domain S^0; by A_k $(k = 1, \dots, n)$ and $B_m (m = 1, \dots, p)$ we denote the vertices (and their affixes) and by $L_0^{(k)}$ and $L_1^{(k)}$ the sides of polygons (A) and (B). By $\pi\alpha_k^0$ and $\pi\beta_m^0$ we denote the sizes of inner angles S^0 at the vertices A_k and B_m, and the angles lying between the ox-axis and exterior normals to the contours L_0 $\left(L_0 = \cup_{k=1}^n L_k^{(0)}\right)$ and L_1 $\left(L_1 = \cup_{m=1}^p L_m^{(1)}\right)$ we denote by $\alpha(t)$ and $\beta(t)$; the positive direction on $L = L_0 \cup L_1$ is taken that which leaves the domain S^0 at the left.

Consider the problem: find the type of the function $z = \omega_0(\zeta)$ conformally mapping the circular ring $D(1 < |\zeta| < R)$ onto the domain S_0.

From the equalities

$$t - A_k = i|t - A_k|e^{i\alpha_k(t)}, \quad t \in L_0^{(k)}; \qquad t - B_m = i|t - B_m|e^{i\beta_m(t)},\ t \in L_1^{(m)},$$

we get

$$\begin{aligned} &\mathrm{Re}\big[t \cdot e^{-i\alpha(t)}\big] = \mathrm{Re}\big[A(t) \cdot e^{-i\alpha(t)}\big], \quad t \in L_0;\\ &\mathrm{Re}\big[t \cdot e^{-i\beta(t)}\big] = \mathrm{Re}\big[B(t) \cdot e^{-i\beta(t)}\big], \quad t \in L_1, \end{aligned} \tag{3}$$

where $A(t)$, $B(t)$, $\alpha(t)$ and $\beta(t)$ are the piecewise constant functions;

$$A(t) = A_k; \qquad \alpha(t) = \alpha_k(t), \quad t \in L_0^{(k)};$$

$$B(t) = B_m, \qquad \beta(t) = \beta_m(t), \quad t \in L_1^{(m)}.$$

From conditions (4) regarding the function $\omega_0(\zeta)$ (after differentiation with respect to the abscissa s), we obtain for the circular ring $\mathcal{D}$ the following Riemann–Hilbert boundary value problem (see [3]):

$$\begin{aligned}&\mathrm{Re}\big[i\sigma\cdot e^{-i\alpha_0(\sigma)}\omega_0'(\sigma)\big]=0,\quad \sigma\in\ell_0\ (|\zeta|=R);\\&\mathrm{Re}\big[i\sigma\cdot e^{-i\beta_0(\sigma)}\omega_0'(\sigma)\big]=0,\quad \sigma\in\ell_1\ (|\zeta|=1);\\&\alpha_0(\sigma)=\alpha[\omega_0(\sigma)];\qquad \beta_0(\sigma)=\beta[\omega_0(\sigma)].\end{aligned}\tag{4}$$

The boundary value problem (5) with respect to the function $\ln\omega_0'(\zeta)$ is reduced in its turn to the Dirichlet problem (1) whose condition of solvability (2) in the class $h(b_1,\dots,b_p)$ (for this class, see [4], §82) has the form

$$\prod_{k=1}^{n}\Big(\frac{a_k}{R}\Big)^{\alpha_k^0-1}\prod_{m=1}^{p}(b_m)^{\beta_m^0-1}=1,$$

(a_k and b_k are the preimages of the points A_k and B_m), and the solution itself of the given class is given by the formula

$$\omega_0'(\zeta)=K_*^0\prod_{j=-\infty}^{\infty}G(R^{2j}\zeta)\,g(R^{2j}\zeta)R^{2\delta_j},$$

where

$$G(\zeta)=\prod_{k=1}^{n}(\zeta-a_k)^{\alpha_k^0-1};\qquad g(\zeta)=\prod_{m=1}^{p}(\zeta-b_m)^{\beta_m^0-1};\qquad \delta_j=\begin{cases}0, & j\ge 0,\\ 1, & j\le -1;\end{cases}$$

K_*^0 is an arbitrary real constant.

Using the above results for the domain S^0 under the condition that (A) is the right n-angle, and (B) is the given rectangle, and considering the domain S as a limiting case S^0 for $n\to\infty$ (in this case $\alpha_0(\sigma)\to\gamma(\sigma)$, where $\gamma(\sigma)=\arg\sigma$, $\sigma\in\ell_0$, $\alpha_k^0\to 1(k=1,2,\dots,n)$), we find that the derivative of the conformally mapping function $z=\omega(\zeta)$ of the circular ring D $(1<|\zeta|<R)$ onto the domain S is a solution of the Riemann–Hilbert problem

$$\mathrm{Re}\big[i\,\omega'(\sigma)\big]=0,\quad \sigma\in\ell_0;\qquad \mathrm{Re}\big[i\,\sigma\,e^{-i\beta_0(\sigma)}\,\omega'(\sigma)\big]=0,\quad \sigma\in\ell_1,\tag{5}$$

and under the condition

$$\prod_{m=1}^{4}(b_m)^{1/2}=1$$

it has the form

$$\omega'(\zeta)=K^0\prod_{m=1}^{4}\Big(1-\frac{b_m}{\zeta}\Big)^{\frac12}\prod_{j=1}^{\infty}\Big(1-\frac{\zeta}{R^{2j}b_m}\Big)^{\frac12}\Big(1-\frac{b_m}{R^{2j}\zeta}\Big)^{\frac12}.\tag{6}$$

(K^0 is an arbitrary real constant).

We get back now to the problem under consideration. On the basis of the well-known Kolosov–Muskhelishvili's formulas (see [5], §41) for finding complex potentials $\varphi(z)$ and $\psi(z)$ in this case we have the boundary conditions

$$\begin{aligned}&\mathrm{Re}\big[\varphi(t)+t\,\overline{\varphi'(t)}+\overline{\psi(t)}\big]=\mathcal{D}_1,\quad t\in L_0,\\&\mathrm{Re}\big[\varkappa\varphi(t)-t\,\overline{\varphi'(t)}-\overline{\psi(t)}\big]=0,\quad t\in L_0,\\&\mathrm{Re}\big[e^{-i\beta(t)}\big(\varphi(t)+t\,\overline{\varphi'(t)}+\overline{\psi(t)}\big)\big]\\&\quad=\mathrm{Re}\bigg[e^{-i\beta(t)}\bigg(i\int_0^s(N(t_0)+iT(t_0))e^{i\beta(t_0)}ds_0+c_1+ic_2\bigg)\bigg],\quad t\in L_1,\\&\mathrm{Re}\big[e^{-i\beta(t)}(\varkappa\varphi(t))-t\,\overline{\varphi'(t)}-\overline{\psi(t)}\big]=2\mu v_n(t),\quad t\in L_1,\end{aligned}$$

where $\varkappa=\frac{\lambda+3\mu}{\lambda+\mu}$ is the Muskhelishvili's constant, λ and μ are the Lame constants, $N(t)$ and $T(t)$ are normal and tangential stresses, respectively.

The obtained conditions in their turn reduce to the two problems

$$\operatorname{Re}\big[\varphi(t)\big] = F_0(t), \quad t \in L_0; \qquad \operatorname{Re}\big[e^{-i\beta(t)}\varphi(t)\big] = F_1(t), \quad t \in L_1; \tag{7}$$

$$\begin{aligned} &\operatorname{Re}\big[\varphi(t) + t\,\overline{\varphi'(t)} + \overline{\psi(t)}\big] = \Gamma_0(t), \quad t \in L_0; \\ &\operatorname{Re}\big[e^{-i\beta(t)}\big(\varphi(t) + t\,\overline{\varphi'(t)} + \overline{\psi(t)}\big)\big] = \Gamma_1(t), \quad t \in L_1, \end{aligned} \tag{8}$$

where

$$F_0(t) = (\varkappa+1)^{-1}D_1, \quad t \in L_0; \qquad F_1(t) = (\varkappa+1)^{-1}\big[c_2 + 2\mu v_n(t)\big], \quad t \in L_1^{(1)};$$
$$F_1(t) = (\varkappa+1)^{-1}(P + c_1), \quad t \in L_2^{(1)};$$
$$F_1(t) = -(\varkappa+1)^{-1}\big[c_2 + 2\mu v_n(t)\big], \quad t \in L_3^{(1)};$$
$$F_1(t) = (\varkappa+1)^{-1}c_1, \quad t \in L_4^{(1)}; \qquad \Gamma_0(t) = D_1, \quad t \in L_0; \qquad \Gamma_1(t) = c_2, \quad t \in L_1^{(1)};$$
$$\Gamma_1(t) = P + c_1, \quad t \in L_2^{(1)}; \qquad \Gamma_1(t) = -c_2, \quad t \in L_3^{(1)}; \qquad \Gamma_1(t) = -c_1, \quad t \in L_4^{(1)}.$$

(D_1, c_1, c_2 are arbitrary real constants, one of which, for example D_1, may be assumed to be zero).

After the domain S is mapped onto the circular ring $D(1 < |\zeta| < R)$, with respect to the function $\varphi_0(\zeta) = \varphi[\omega(\zeta)]$, from (9) we obtain the Riemann–Hilbert boundary value problem for the circular ring

$$\operatorname{Re}\big[\varphi_0(\sigma)\big] = F_{00}(\sigma), \quad \sigma \in \ell_0; \qquad \operatorname{Re}\big[e^{-i\beta_0(\sigma)}\varphi_0(\sigma)\big] = F_{10}(\sigma), \quad \sigma \in \ell_1, \tag{9}$$

where

$$F_{00}(\sigma) = F_0[\omega(\sigma)], \quad \sigma \in \ell_0; \qquad F_{10}(\sigma) = F_1[\omega(\sigma)], \quad \sigma \in \ell_1; \qquad \beta_0(\sigma) = \beta[\omega(\sigma)].$$

Let us consider both the homogeneous problem corresponding to problem (9),

$$\operatorname{Re}\big[\varphi_0(\sigma)\big] = 0, \quad \sigma \in \ell_0; \qquad \operatorname{Re}\big[e^{-i\beta_0(\sigma)}\varphi_0(\sigma)\big] = 0, \quad \sigma \in \ell_1,$$

and the auxiliary problem

$$\operatorname{Re}\big[i\sigma\,\chi_0(\sigma)\big] = 0, \quad \sigma \in \ell_0; \qquad \operatorname{Re}\big[i\sigma\,e^{-i\beta_0(\sigma)}\chi_0(\sigma)\big] = 0, \quad \sigma \in \ell_1. \tag{10}$$

We will seek for solutions of that problem of the class $h(b_1, \dots, b_4)$. The index of problem (10) of that class equals -2.

Taking now into account that the function

$$T(\zeta) = \Big(1 - \frac{\zeta}{R}\Big)^{-2} \prod_{j=1}^{\infty} \Big(1 - \frac{\zeta}{R \cdot R^{2j}}\Big)^{-2} \Big(1 - \frac{R}{R^{2j}\zeta}\Big)^{-2}$$

satisfies the conditions

$$\frac{T(\sigma)}{\overline{T(\sigma)}} = \frac{\overline{\sigma}}{\sigma}, \quad \sigma \in \ell_0; \qquad \frac{T(\sigma)}{\overline{T(\sigma)}} = 1, \quad \sigma \in \ell_1,$$

and writing conditions (8) in an expanded form

$$\omega'(\sigma) - \overline{\omega'(\sigma)} = 0, \quad \sigma \in \ell_0; \qquad \omega'(\sigma) - \frac{\overline{\sigma}}{\sigma}\,e^{2i\beta_0(\sigma)}\overline{\omega'(\sigma)} = 0, \quad \sigma \in \ell_1,$$

we conclude that the function $\chi_0(\zeta)$ can be represented by the formula

$$\chi_0(\zeta) = K_1 T(\zeta)\,\omega'(\zeta)$$

(K_1 is an arbitrary real constant).

Thus with respect to the function $\Theta(\zeta) = K_1\zeta\,T(\zeta)\,\omega'(\zeta)$, we obtain the equalities

$$\frac{\Theta(\sigma)}{\overline{\Theta(\sigma)}} = 1, \quad \sigma \in \ell_0; \qquad \frac{\Theta(\sigma)}{\overline{\Theta(\sigma)}} = e^{2i\beta_0(\sigma)}, \quad \sigma \in \ell_1$$

and, hence, boundary conditions (9) can be rewritten in the form

$$\begin{aligned}
&\mathrm{Re}\left[\frac{\varphi_0(\sigma)}{\sigma\,T(\sigma)\,\omega'(\sigma)}\right]=\frac{F_{00}(\sigma)}{\sigma\,T(\sigma)\,\omega'(\sigma)},\qquad \sigma\in\ell_0;\\
&\mathrm{Re}\left[\frac{\varphi_0(\sigma)}{\sigma\,T(\sigma)\,\omega'(\sigma)}\right]=\frac{F_{10}(\sigma)\,e^{i\beta_0(\sigma)}}{\sigma\,T(\sigma)\,\omega'(\sigma)},\qquad \sigma\in\ell_1.
\end{aligned}\tag{11}$$

The condition of solvability of problem (11) has the form (see item 1):

$$\int_{\ell_0}\frac{F_{00}(\sigma)}{\sigma\,T(\sigma)\,\omega'(\sigma)}\,\frac{d\sigma}{\sigma}=\int_{\ell_1}\frac{F_{10}(\sigma)\,e^{i\beta_0(\sigma)}}{\sigma\,T(\sigma)\,\omega'(\sigma)}\,\frac{d\sigma}{\sigma}\tag{12}$$

and the solution itself is given by the formula

$$\varphi_0(\zeta)=\zeta\,T(\zeta)\,\omega'(\zeta)\,M(\zeta),$$

where

$$M(\zeta)=\frac{1}{\pi i}\sum_{j=-\infty}^{\infty}\left[\int_{\ell_0}\frac{F_{00}(\sigma)d\sigma}{(\sigma-R^{2j}\zeta)\sigma\,T(\sigma)\,\omega'(\sigma)}+\int_{\ell_1}\frac{F_{10}(\sigma)\,e^{i\beta_0(\sigma)}d\sigma}{(\sigma-R^{2j}\zeta)\sigma\,T(\sigma)\,\omega'(\sigma)}\right]+E_0+iE_1,\tag{13}$$

$$E_0=-\frac{1}{2\pi i}\int_{\ell_0}\frac{F_{00}(\sigma)}{\sigma\,T(\sigma)\,\omega'(\sigma)}\,\frac{s\sigma}{\sigma};$$

E_1 is an arbitrary real constant.

Taking into account that the function $T(\zeta)$ at the point $\zeta=R$ has the pole of the second order, we conclude that for the function $\varphi_0(\zeta)$ to be continuously extendable in the domain $D\cup\ell$, it is necessary and sufficient that the conditions

$$M(R)=0;\qquad M'(R)=0\tag{14}$$

be fulfilled.

Denoting $M_0(\zeta)=T(\zeta)\,M(\zeta)$ and taking into account that $\varphi(z)=\varphi[\omega(\zeta)]=\varphi_0(\zeta)$, and hence $\varphi'(z)=\varphi_0'(\zeta)[\omega'(\zeta)]^{-1}$, we obtain

$$\varphi'(z)=\frac{\varphi_0'(\zeta)}{\omega'(\zeta)}=M_0(\zeta)+\zeta\,\frac{\omega''(\zeta)}{\omega'(\zeta)}\,M_0(\zeta)+\zeta\,M_0'(\zeta).\tag{15}$$

On the basis of the results given in [4] (§26) concerning the behavior of the Cauchy type integral in the vicinity of points of density discontinuity, we can conclude that in the vicinity of the point $b_k (k=1,\dots,4)$ the function $M_0(\zeta)$ can be represented in the form

$$M_0(\zeta)=\frac{k}{(\zeta-b_k)^{1/2}}+\Omega_k^0(\zeta),\qquad k=1,\dots,4,$$

where k is the definite constant and $\Omega_k^0(\zeta)$ in the vicinity of the point b_k admits the estimate

$$\left|\Omega_0^{(k)}(\zeta)\right|<\frac{C}{|(\zeta-b_k)^{\alpha_0}|},\qquad C=\mathrm{const},\ 0<\alpha_0<1/2.$$

As is known (see [6], §37), for the conformally mapping function $\omega(\zeta)$ in the vicinity of angular points the estimates

$$\begin{aligned}
&\omega(\zeta)=B_k+(\zeta-b_k)^{\beta_k^0}\,\Omega_k(\zeta),\\
&\zeta\,\frac{\omega''(\zeta)}{\omega'(\zeta)}=\frac{b_k(\beta_k^0-1)}{\zeta-b_k}+\Omega_k^*(\zeta),\qquad k=1,\dots,4,
\end{aligned}$$

hold, where $\Omega_k(b_k)\neq 0$, $\Omega_k^*(\zeta)$ is the right part of the Laurent decomposition of the function $\zeta\,\frac{\omega''(\zeta)}{\omega'(\zeta)}$.

Taking the above-said into account, from (15) we obtain the estimate

$$\varphi'(z) = \frac{K_0}{(\zeta - b_k)^{1/2}} + Q_0^k(\zeta), \quad k = 1, \ldots, 4, \; K_0 = \frac{1}{2} K,$$

and hence, in the vicinity of the point B (B is one of the points $B_k (k = 1, \ldots, 4)$), we have the estimates

$$\left|\varphi'(z)\right| < M_1 \left|z - B\right|^{-1/3}; \qquad \left|\varphi''(z)\right| < M_2 \left|z - B\right|^{-4/3}; \quad M_1, M_2 = \text{const}. \tag{16}$$

After the function $\varphi(z)$ is defined, the finding of the function $\psi(z)$ by virtue of (8) reduces to the problem, analogous to problem (7),

$$\operatorname{Re}\left[R(t)\right] = N_0(t), \quad t \in L_0; \qquad \operatorname{Re}\left[e^{i\beta(t)} R(t)\right] = N_1(t), \quad t \in L_1, \tag{17}$$

where

$$R(z) = \psi(z) + P(z)\,\varphi'(z);$$
$$N_0(t) = \Gamma_0(t) - \operatorname{Re}\left[(\overline{\varphi(t)}) + (\bar{t} - P(t))\varphi'(t)\right], \quad t \in L_0;$$
$$N_1(t) = \Gamma_1(t) - \operatorname{Re}\left[e^{i\beta(t)}(\overline{\varphi(t)}) + (\bar{t} - P(t))\varphi'(t)\right], \quad t \in L_1,$$

$P(z)$ is the interpolated polynomial satisfying the condition $P(B_k) = \overline{B_k} (k = 1, \ldots, 4)$ and having the form

$$P(z) = \frac{(z - B_2)\cdots(z - B_4)}{(B_1 - B_2)\cdots(B_1 - B_4)} \cdot \overline{B_1} + \cdots + \frac{(z - B_1)\cdots(z - B_3)}{(B_4 - B_1)\cdots(B_4 - B_3)} \cdot \overline{B_4}.$$

Insertion of the polynomial $P(z)$ into consideration ensures the boundedness of the right-hand side in the boundary condition (17), and thus, a solution of that problem can be constructed analogously to the previous one (see problem (7)), namely, after the domain D is conformally mapped on S, the factorization of problem (14) is written as follows:

$$1 = \frac{\overline{\sigma}\,\overline{\omega'(\sigma)}\,\overline{T(\sigma)}}{\sigma\,\omega'(\sigma)\,T(\sigma)}, \quad \sigma \in \ell_0; \qquad e^{-2i\beta_0(\sigma)} = \frac{\overline{\sigma}\,\overline{\omega'(\sigma)}\,\overline{T(\sigma)}}{\sigma\,\omega'(\sigma)\,T(\sigma)}, \quad \sigma \in \ell_1,$$

and the condition of solvability will have the form

$$\int_{\ell_0} N_0(t)\,\omega'(t)\,T(t)dt = \int_{\ell_1} N_1(t)\,e^{-i\beta(t)}\omega'(t)\,T(t)dt. \tag{18}$$

If this condition is fulfilled, the solution of problem (17) is represented by the formula

$$R_0(\zeta) = \left[\zeta\,T(\zeta)\,\omega'(\zeta)\right]^{-1} M_1(\zeta),$$

where $R_0(\zeta) = R[\omega(\zeta)]$, and the function $M_1(\zeta)$ having the form, analogous to the function $M(\zeta)$ (see formula (13)) and involving one arbitrary real constant E_2, can be easily written out.

The condition of continuous extendability of the function $R_0(\zeta)$ in the domain $D \cup \ell$ has the form

$$M_1(b_k) = 0, \quad k = 1, \ldots, 4, \tag{19}$$

and thus we have eight conditions (conditions (12), (14), (18) and (19)) with respect to eight real constants: C_1, C_2, E_1, E_2, $v_n^{(k)}(t) (k = 1, \ldots, 4)$. These conditions are, in fact, a system of linear algebraic equations with real coefficients of the form

$$\sum_{k=1}^{8} a_{ik}\,d_k = \ell_i \quad (i = 1, \ldots, 8),$$

where $a_{ik} (i, k = 1, \ldots, 8)$ are the known real constants, independent of the external forces P, and ℓ_i are the constants vanishing for $P = 0$. d_k are the above-mentioned unknown real constants $(c_1, c_2, \ldots)$.

Assuming that the determinant of the system equals zero, we find that the homogeneous system

$$\sum_{k=1}^{8} a_{ik}\,d_k = 0, \quad i = 1, \ldots, 8,$$

has nontrivial solutions which as a consequence implies that the problem under consideration has a solution (different from a rigid displacement), representable by complex potentials $\varphi_1(z)$ and $\psi_1(z)$ for which the equality (see [4], §113)

$$\operatorname{Im}\int_L \left[\overline{\varphi_1(t)}+\bar{t}\,\varphi_1'(t)+\psi_1(t)\right]d\left[\varkappa\varphi_1(t)-t\,\overline{\varphi_1'(t)}-\overline{\psi_1(t)}\right]$$
$$=4\iint_S\left\{2(\varkappa-1)\operatorname{Re}\left[\varphi_1'(z)\right]^2+\left|\bar{z}\,\varphi_1''(z)+\psi_1'(z)\right|^2\right\}dx\,dy,$$

is valid. In our case this equality can be rewritten in the form

$$\int_L\left\{\operatorname{Re}\, i\,e^{-i\nu(t)}\left[\varphi_1(t)+t\,\overline{\varphi_1'(t)}+\overline{\psi_1(t)}\right]d\operatorname{Re}\,e^{i\nu(t)}\left[\varkappa\varphi_1(t)-t\,\overline{\varphi_1'(t)}-\overline{\psi_1(t)}\right]\right\}$$
$$=4\iint_S\left\{2(\varkappa-1)\left[\operatorname{Re}\varphi_1'(t)\right]^2+\left|\bar{z}\,\varphi_1''(z)+\psi_1'(z)\right|^2\right\}dx\,dy. \tag{20}$$

When writing this equality, we have taken into account continuous extensions of the expression $\varkappa\varphi_1(z)-z\,\overline{\varphi_1'(z)}-\overline{\psi_1(z)}$ in the domain $S\cup L$ and those of $\varphi_1'(z)$ and $\bar{z}\,\varphi_1''(z)+\psi_1'(z)$ up to the boundary L, except possibly the points $B_k(k=1,\ldots,4)$.

Taking into account the boundary conditions $(*)$, from (20) we have

$$\varphi_1'(z)=i\,H_0,\qquad \varphi_1(z)=i\,H_0z+H_1;\qquad \psi_1(z)=H_2,$$

where H_0 is the real and H_1 and H_2 are the complex constants, and thus we have rigid displacement of the body as a whole which contradicts our assumption, and consequently, the system determinant is different from zero and the above-posed problem is uniquely solvable.

Remark. The obtained results can be extended to the case of a circular domain with a rectilinear cut (the cut can be considered as a limiting case of a rectangle under contraction of segments B_2B_3 and B_4B_1 to a point). In this case the conformally mapping function has the form

$$\omega'(\zeta)=K^{00}\prod_{k=1}^{2}\left(1-\frac{b_k}{\zeta}\right)\prod_{j=1}^{\infty}\prod_{k=1}^{2}\left(1-\frac{\zeta}{R^{2j}b_k}\right)\left(1-\frac{b_k}{R^{2j}\zeta}\right),$$

and the estimate (23) can be written as follows:

$$|\varphi'(z)|<M_1|z-B_k|^{-1/2};\qquad |\varphi''(z)|<M_2|z-B_k|^{-3/2};\quad k=1,2.$$

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